Mathematics

Practical problems leading to differential equations

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Content

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Introduction

Differential equations occur in a variety of areas in applied mathematics, physics, mechanics etc. They are a valuable instrument in enabling description of nearly any problems involving dynamics of machines and mechanisms.

Mathematical description of physical laws (and chiefly of the fundamental laws of physics, i.e. those which lie in the basis of our understanding of processes taking place in nature) is most commonly facilitated by differential equations. Hence, they are of great significance to modern-day technology; especially those in which physics plays a leading role [2].

Target of the project

The target of this project is attempting to construct mathematical models in the format of differential equations. As the material for the creation of the aforementioned models one can use several applicable problems.

We will use these mathematical models in order to analyse two concepts; namely, the trajectory of a moth's flight and the change in velocity of a rocket during its motion.

Problem about the flight of a moth

Introduction

Recent studies suggest moths use the Moon to enable them to navigate through the dark. When a moth wants to fly from point A to point G, to ensure flight in a straight line the moth "measures" the angle φ between AM and AG – namely, the angle between the shortest path to the moon and the direction of flight of the moth (Fig. 1). For the moth to continue travelling in a straight line, the angle φ has to be kept constant i.e. for the moth to view the moon in periphery in a fixed way [3].

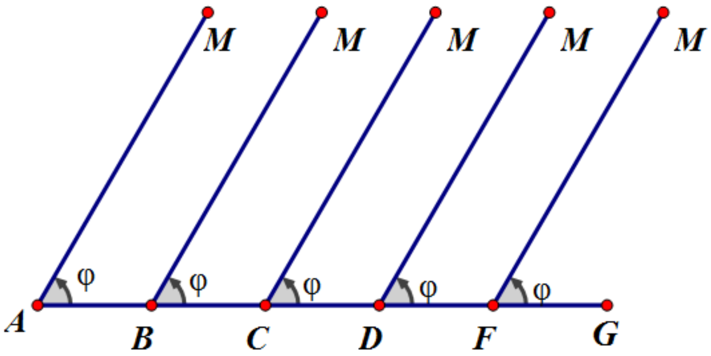

Fig. 1

This property of the moth can be attributed to its unique eye structure, which consists of a number of so-called "ommatidia" which act like a mini-eye enabling a very wide scope of view for the moth of a stunning 270 degrees. Each ommatidium is capable of receiving light from only one direction, therefore to keep its head "fixed" – thus flying in a straight direction – the moth has to keep the Moon in the field of view of the same ommatidia, hence meaning its orientation stays constant (Fig. 2).

Fig. 2

The moon can be viewed as an infinitely distant source of light, hence all the paths from the moth to the moon (AM,BM,CM…) are parallel to each other; once again due to the fact that the moon is infinitely distant from the moth, and hence a small change in distance leads to a negligible change in φ. Thus, a straight route is achieved by the moth.

The lamp, on the other hand, is situated in a finite distance from the moth therefore the direction to the lamp continuously changes when the moth "mistakes" the lamp for the moon, therefore keeping its direction and φ constant. (Fig. 3).

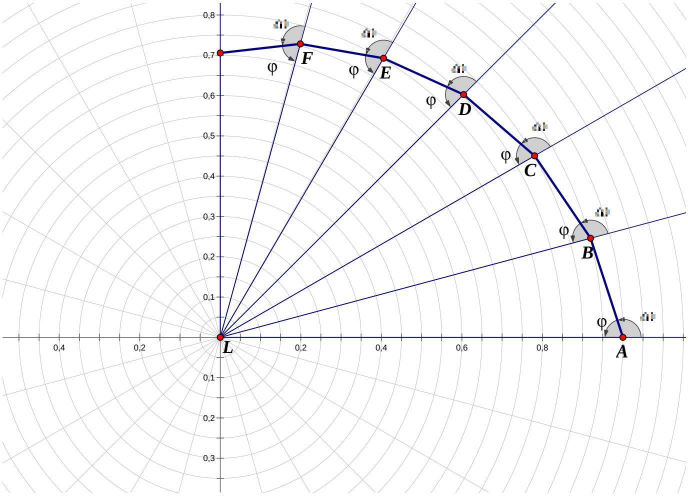

Fig. 3

If the moth would correct its flight trajectory at points B,C…, its flight path would be slightly jagged. In actual fact, the direction towards the lamp is constantly changing forcing the moth to constantly correct its direction of flight. The result is that the moth's direction of flight can be depicted as a smooth curve. (Fig. 4).

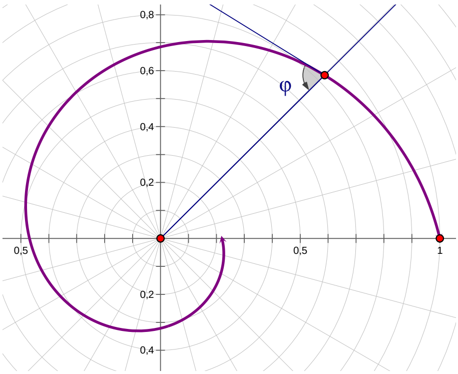

Fig. 4

Formulating the problem. Find an equation of the curve which a moth flies by if it mistakes the lamp for the Moon.

In order to solve this problem, we will introduce a slightly different view on the coordinate system. Instead of using Cartesian coordinates, we will be using polar coordinates.

The polar coordinate system

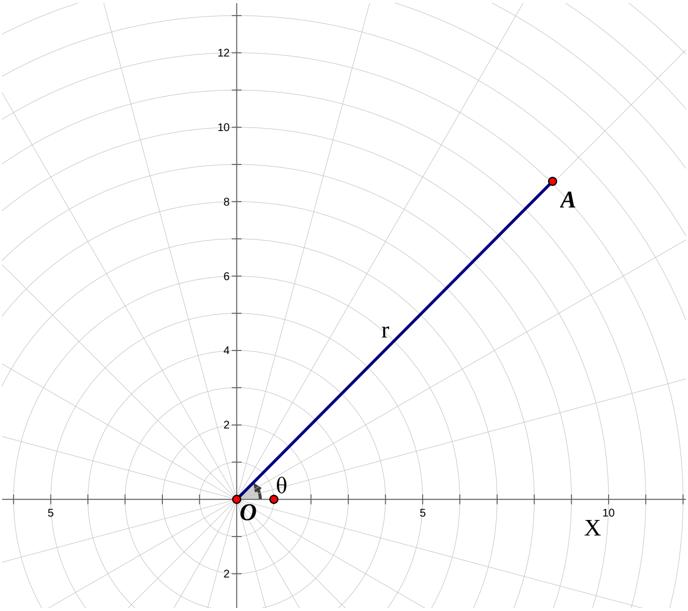

Fig. 5

We can determine the position of point A when we set up the coordinate system if we find the distance r to point A and the angle θ between vector $\overline{OA}$ and OX (x-axis, Fig 5).

Let us clarify the relationship between the polar coordinate system (r, θ) and the Cartesian coordinate system (x, y)(Fig.6).

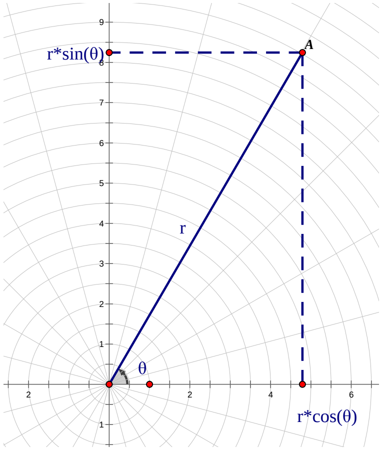

Fig. 6

$$\begin{cases} x = r(\theta) * \cos\theta \\ y = r(\theta) * \sin\theta \end{cases} \quad (1)$$

Tangent to a curve in polar coordinates

In the Cartesian coordinate system, the geometrical understanding of the derivative is that the tangent of angle δ (Fig. 7) – the angle between the tangent and the *x*-axis – at point $A(x_0, f(x_0))$ is equivalent to the derivative at that point.

$$\tan(\delta) = f'(x_0) \quad (2)$$

Let us attempt to achieve a similar result in the polar coordinate system.

For the polar coordinate system, it is simpler to consider the angle ω between the velocity and the continuation of vector $\overline{\mathrm{OA}}$ as a pose to the angle δ between OX (x-axis) and the tangent (velocity). (Fig. 7).

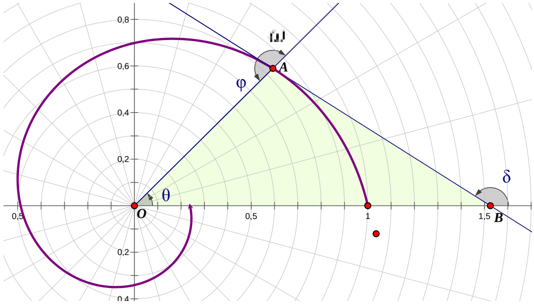

Fig. 7

Using the exterior angle theorem (exterior angle = sum of two opposite interior angles), the exterior angle of δ of triangle BAO is equal to:

$$\delta = \omega + \theta, \quad (3)$$

$$\omega = \delta - \theta \quad (4)$$

Let us find tan(ω) :

$$\tan(\omega) = \tan(\delta - \theta) = \frac{\tan\delta - \tan\theta}{1 + \tan\delta \cdot \tan\theta} = \frac{y'_x - \frac{y}{x}}{1 + y'_x * \frac{y}{x}} =$$

$$= \frac{x * y'_x - y}{x + y'_x y} \quad (5)$$

Calculate y'_x, when changing to polar coordinates:

$$y'_x = \frac{dy}{dx} = \frac{dy}{d\theta} * \frac{d\theta}{dx} = \frac{dy}{d\theta} : \frac{dx}{d\theta} = \frac{y'_\theta}{x'_\theta} \quad (6)$$

Using formula (1) and the product rule, we receive:

$$y'_\theta = r'(\theta) * \sin\theta + r(\theta) * \cos\theta, \quad (7)$$

$$x'_\theta = r'(\theta) * \cos\theta - r(\theta) * \sin\theta, \quad (8)$$

Substituting formula (7) and (8) into formula (6), we get:

$$y'_x = \frac{r'(\theta) * \sin\theta + r(\theta) * \cos\theta}{r'(\theta) * \cos\theta - r(\theta) * \sin\theta} \quad (9)$$

In order to simplify formula (5), we will split the formula up into several fragments:

$$x * y'_x = \frac{(r'(\theta) * \sin\theta + r(\theta) * \cos\theta)r(\theta) * \cos\theta}{r'(\theta) * \cos\theta - r(\theta) * \sin\theta} =$$

$$= \frac{rr' \sin\theta \cos\theta + r^2 \cos^2\theta}{r' \cos\theta - rsin\,\theta} \quad (10)$$

Hence, the numerator (formula 5) takes the form:

$$x * y'_x - y = \frac{rr' \sin\theta \cos\theta + r^2 \cos^2\theta}{r' \cos\theta - rsin\,\theta} -$$

$$- \frac{r \sin\theta\,(r' \cos\theta - rsin\,\theta)}{r' \cos\theta - rsin\,\theta} = \frac{r^2}{r' \cos\theta - rsin\,\theta} \quad (11)$$

And the denominator of the fraction in formula (5) takes the form:

$$x + y'_x y = x + \frac{(r' \sin\theta + r \cos\theta) r \sin\theta}{r' * \cos\theta - r \sin\theta} =$$

$$= \frac{r * \cos\theta\,(r' \cos\theta - r \sin\theta) + (r' \sin\theta + r \cos\theta) r \sin\theta}{r' \cos\theta - r \sin\theta} =$$

$$= \frac{rr' \cos^2\theta + rr' \sin^2\theta}{r' \cos\theta - r \sin\theta} = \frac{rr'}{r' \cos\theta - r \sin\theta} \quad (12)$$

Substituting formula (11) and (12) into (5), the final outcome becomes:

$$\tan\omega = \frac{x * y'_x - y}{x + y'_x y} = \frac{r^2}{r' \cos\theta - rsin\,\theta} : \frac{rr'}{r' \cos\theta - r \sin\theta} =$$

$$= \frac{r^2}{rr'} = \frac{r}{r'} \quad (13)$$

Therefore, we receive an identity for a tangent to a curve in the polar coordinate system. The tangent of angle ω, formed from the continuation of OA i.e the radius vector (Fig.7) can now be formulated using the simple identity:

$$\tan \omega = \frac{r}{r'} \quad (14)$$

Notice that formulae from (5) to (14) are mathematically valid if $r \neq 0$ and $r'_{\theta} \neq 0$.

It is unlikely that $r = 0$ because it would mean the moth is in the lamp.

If $r'_{\theta} = 0$ – in other words, $r = const$ – our curve would be defined as a circumference of radius r. This would mean that the moth flies in a circular path and would never reach the lamp.

Formula (14) is discussed in book [5].

As the butterfly always tries to keep the angle ω constant, we receive a separable, first-order differential equation:

$$\frac{r}{r'} = \tan \omega \,; \qquad \frac{r'}{r} = \cot \omega \quad (15)$$

$$\frac{dr}{d\theta \cdot r} = \cot \omega \quad (16)$$

$$\frac{dr}{r} = \cot \omega \, d\theta \quad (17)$$

$$\int \frac{dr}{r} = \int \cot \omega \, d\theta \quad (18)$$

$$\ln r = \cot\omega \cdot \theta + C \quad (19)$$

$$r(\theta) = e^{\cot\omega\cdot\theta + C} = e^{C} \cdot e^{\cot\omega\cdot\theta} \quad (20)$$

Let the initial distance (at θ=0) with between the moth and origin (the lamp) be equal to A.

Solving for initial conditions, we get:

$$r(0) = e^{C} * e^{0} = e^{C} = A \quad (21)$$

$$r(\theta) = A \cdot e^{\cot\omega\cdot\theta} \quad (22)$$

We can depict the trajectory of the butterfly's flight in polar coordinates as follows (Fig.8):

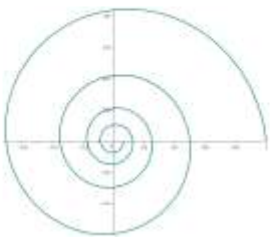

Fig. 8

Conclusion.

If the angle between the velocity and the direction to the lamp is 90 degrees, the moth flies in a circular motion around a circumference as we said before (Fig.9). Any way, formula (22) remains true even if $\omega = \frac{\pi}{2}$. (Substitute $\omega = \frac{\pi}{2}$ into formula (20)):

$$r(\theta) = A \cdot e^{\cot\frac{\pi}{2}\cdot\theta} = A = const \quad (23)$$

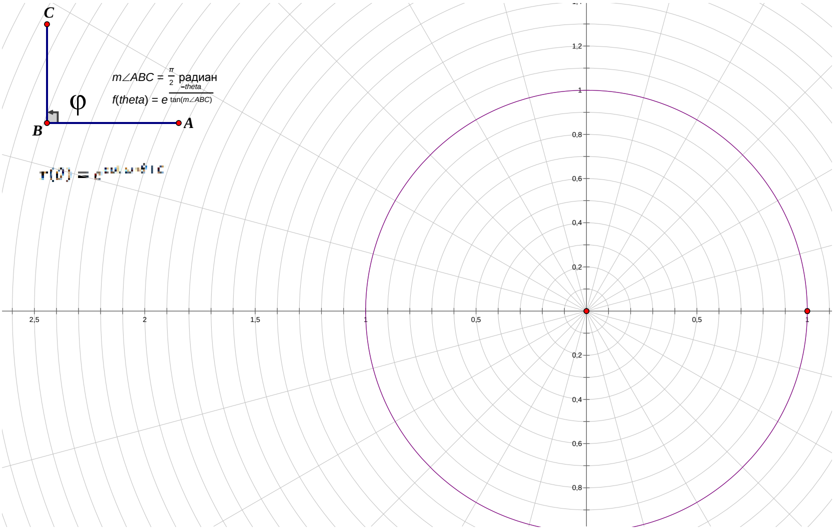

Fig. 9

If the angle between the velocity of the moth and the direction to the lamp is zero, the moth flies in a straight line to the origin.

If the angle between the velocity of the moth and the direction to the lamp is acute, then with every rotation around the lamp the moth begins spiralling in, closer and closer to the lamp, until it collides with the lamp and subsequently hurts itself, however if the angle is obtuse, with every rotation the butterfly will distance itself from the lamp.

A precaution for the reader is that you might be tempted to believe a moth lacks intelligence as it considers the lamp as the moon. In actuality, pilots in aeroplanes and captains of ship crews operate alike the moth, by deciding on a "straight" direction to travel in when they orienteer using a compass (ships) or a radio direction finder(pilots). Both aeroplanes and ships travel in small sections of logarithmic spirals (Fig.8).

Problem investigating the flight of a rocket

Setting the scene

The mass of a rocket with a full fuel tank is equal to M, whilst an empty one would be m. The speed at which the combustion products leave the rocket is c, the initial speed of the rocket is zero. Find the speed of the rocket after all the fuel has been burnt, not taking into account gravitational force and air resistance (Tsiolkovsky formula) [4].

Fig. 10

Forming a mathematical model

Let us consider the motion of the rocket along a straight line directed vertically, with the origin of the coordinate system being located at the initial position of the rocket (Figure 11). Let M(t) be the mass of rocket at moment t, with the mass of the rocket continuously changing as the fuel is being burnt. Let v(t) be the velocity of the rocket at moment t, with initial velocity being zero ($v(0)=0$). Then, the momentum of the rocket k(t) = M(t)*v(t).

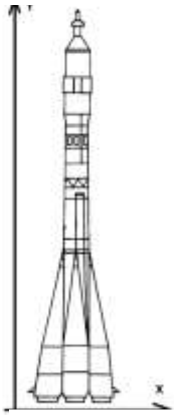

Fig. 11

From that, due to the conservation of momentum we can equate it to

$$\mathrm{k}(\mathrm{t}+\Delta\mathrm{t})-\mathrm{k}(\mathrm{t})=\Delta\mathrm{p}(\mathrm{t}) \quad (24)$$

where $p(t)$ is the momentum of external forces which act for a time of Δt on a time interval of $t+\Delta t$.

The momentum of the external forces is created by the products of the combustion of fuel, which leave the rocket at a speed[1] of $c–v(t)$, where c is the relative speed of the combustion products leaving the rocket.

As, during this time interval, the mass also decreases – from $M(t)$ to $M((t+\Delta\mathrm{t})$- we receive:

$$\Delta p=(c-v(t))\big(M(t)-M(t+\Delta t)\big) \quad (25)$$

Let us now regard the left hand side of equation (24).

$$\mathrm{k}(\mathrm{t}+\Delta\mathrm{t})-\mathrm{k}(\mathrm{t})=M(t+\Delta t)v(t+\Delta t)-M(t)v(t) \quad (26)$$

Substituting formula (25) and (26) into (24), we get:

$$M(t+\Delta t)v(t+\Delta t)-M(t)v(t)=\big(c-v(t)\big)(M(t)-M(t+\Delta t)) \quad (27)$$

After simple algebraic manipulation of formula (27), the equation can be written as:

$$\begin{aligned}&[M(t+\Delta t)*v(t+\Delta t)-M(t+\Delta t)*v(t)]\\&\qquad+[M(t+\Delta t)*v(t)-M(t)v(t)]=\\&\qquad=\big(c-v(t)\big)(M(t)-M(t+\Delta t)) \quad (28)\end{aligned}$$

$$\big[M(t+\Delta t)\big(v(t+\Delta t)-v(t)\big)\big]+[v(t)(\Delta M)]=-\big(c-v(t)\big)\Delta M \quad (29)$$

$$M(t+\Delta t)\big(v(t+\Delta t)-v(t)\big)+v(t)(\Delta M)=-c\Delta M+v(t)\Delta M \quad (30)$$

[1] c-v –это скорость продуктов горения относительно Земли

$$M(t+\Delta t)\big(v(t+\Delta t)-v(t)\big)=-c\Delta M \quad (31)$$

$$M(t+\Delta t)(\Delta v)=-c\Delta M \quad (32)$$

Note that we can substitute the left hand side of equation (32) as:

$$M(t+\Delta t)(\Delta v)=M(t)\Delta v+o(\Delta v) \quad (33)$$

Because

$$M(t+\Delta t)(\Delta v)-M(t)\Delta v=o(\Delta v) \quad (34)$$

$$\Delta v\Delta M=o(\Delta v) \quad (35)$$

$$\Delta M=\frac{o(\Delta v)}{\Delta v}\to 0 \quad (36)$$

Using formula (33), let us change the form of (32):

$$M(t)\Delta v+o(\Delta v)=-c\Delta M \quad (37)$$

Then

$$M(t)+\frac{o(\Delta v)}{\Delta v}=-\frac{c\Delta M}{\Delta v} \quad (38)$$

Now we can set a limit to $\Delta v \to 0$:

$$M(t)=-c\frac{dM}{dv} \quad (39)$$

Now we can regard the mass of the rocket not only in terms of time, but in terms of velocity:

$$M(v)=-c\frac{dM}{dv} \quad (40)$$

Solving a differential equation with separable variables:

$$-\frac{1}{c}dv = \frac{dM}{M} \quad (41)$$

Integrating, we receive:

$$-\frac{1}{c}\int dv = \int \frac{dM}{M} \quad (42)$$

$$c_1 - \frac{1}{c}v = \ln M \quad (43)$$

$$\frac{1}{c}v = -\ln M + c_1 \quad (44)$$

$$\frac{1}{c}v = \ln\left(\frac{1}{M}\right) + c_1 \quad (45)$$

$$v(M) = c \cdot \ln\left(\frac{1}{M}\right) + c_0 \quad (46)$$

Now, let's view the mass of a rocket as a function of time once again:

$$v(t) = c \cdot \ln\left(\frac{1}{M(t)}\right) + c_0 \quad (47)$$

Using the fact that the initial velocity of a rocket i.e. at the start is 0, we can solve for initial conditions:

$$0 = v(0) = c \cdot \ln\left(\frac{1}{M(0)}\right) + c_0 \quad (48)$$

$$c_0 = -c \cdot \ln\left(\frac{1}{M}\right) \quad (49)$$

Substituing (49) into (46):

$$v(t) = c \cdot \ln\left(\frac{1}{M(t)}\right) - c \cdot \ln\left(\frac{1}{M}\right) = c\left(\ln\left(\frac{1}{M(t)} : \frac{1}{M}\right)\right) = c \cdot \ln\frac{M}{M(t)} \quad (50)$$

Overall, we receive:

$$v(t) = c \cdot \ln \frac{M}{M(t)} \quad (51)$$

Conclusion.

If one supposes that the rate of fuel combustion is constant and the fuel supplies end at time t_0, the the graph of velocity $v(t)$ can be presented as Fig. 12.

Fig. 12

If t_0 be the moment in time, when all the rocket's fuel has been burnt ($M(t_0)$), then

$$v_{max} = c \cdot \ln \frac{M}{m} \quad (Tsiolkovsky's\ formula)(52)$$

If x is the mass of combusted fuel, then

$$v(x) = c * \ln \frac{M}{M - x} \quad (53)$$

Formula (53) was derived by Russian scientist Konstantin Tsiolkovsky[2] who independently derived it and published it in his 1903 work.[6] The equation had been derived earlier by the British mathematician William Moore in 1810, and later published in a separate book in 1813. The minister William Leitch, who was a capable scientist, also independently derived the fundamentals of rocketry in 1861.

In Fig. 13, you can see a Belarusian post stamp honoring the 45th anivessary of space exploration.

Fig. 13

[2] [1]

Future development of the project

- Construct a model which investigates the changing velocity of a multistage rocket.
- Expand the scope of problems which we can attempt to create mathematical models for and thus implementing differential equations in their solution.

Resources

- In order to create illustrations for our investigations we used several mathematical applications: Maple and Geometrical Sketchpad.

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